Basic Fourier series: convergence on and outside the q-linear grid

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Abstract. A *q-type Hölder condition* on a function f is given in order to establish (uniform) convergence of the corresponding basic Fourier series $S_q[f]$ to the function itself, on the set of points of the q-linear grid.

Furthermore, by adding others conditions, one guaranties the (uniform) convergence of $S_q[f]$ to f on and "outside" the set points of the q-linear grid.

Key words and phrases: q-trigonometric functions, q-Fourier series, Basic Fourier expansions, uniform convergence, q-linear grid.

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1. Introduction

Basic Fourier expansions on q-quadratic and on q-linear grids were first considered in [**8**] and in [**7**], respectively. Recently, in [**10**], sufficient conditions for (uniform) convergence of the q-Fourier series in terms of basic trigonometric functions S_q and C_q, on a q-linear grid, were given. In [**19**] it was established an "addition" theorem for the corresponding basic exponential function, being these functions equivalent to the ones introduced by H. Exton in [**12**]. Following the unified approach of M. Rahman in [**18**], these functions can be seen as analytic linearly independent solutions of the initial value problem

$$\frac{\delta f(x)}{\delta x} = \lambda f(x)\,, \quad f(0) = 1\,,$$

where δ is the symmetric q-difference operator acting on a function f by

$$\delta f(x) = f(q^{1/2}x) - f(q^{-1/2}x)\,, \tag{1.1}$$

with $0 < q < 1$. Then, from (1.1),

$$\frac{\delta f(x)}{\delta x} = \frac{f(q^{1/2}x) - f(q^{-1/2}x)}{x(q^{1/2} - q^{-1/2})}\,. \tag{1.2}$$

There exists an important relation between this difference operator and the q-integral. The q-integral is defined by

$$\int_0^a f(x)d_qx = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^b f(x)d_qx\,. \tag{1.3}$$

From (1.2) and (1.3) it follows

$$\int_{-1}^1 \frac{\delta f(x)}{\delta x}d_qx = q^{\frac{1}{2}}\left\{\left[f(q^{-\frac{1}{2}}) - f(-q^{-\frac{1}{2}})\right] - \left[f(0^+) - f(0^-)\right]\right\}\,, \tag{1.4}$$

hence, one have the following formula [**10**] for q-integration by parts:

$$\begin{aligned} \int_{-1}^1 g\big(q^{\pm\frac{1}{2}}x\big)\frac{\delta_q f(x)}{\delta_q x}d_qx \;&=\; -\int_{-1}^1 f\big(q^{\mp\frac{1}{2}}x\big)\frac{\delta_q g(x)}{\delta_q x}d_qx\, + \\ q^{\frac{1}{2}}\left\{\Big[\,(fg)\big(q^{-\frac{1}{2}}\big) - (fg)\big(-q^{-\frac{1}{2}}\big)\,\Big] - \Big[\,(fg)\big(0^+\big) - (fg)\big(0^-\big)\,\Big]\right\}. \end{aligned} \tag{1.5}$$

These functions satisfy an orthogonality relation [**7, 12**] where the corresponding inner product is defined in terms of the q-integral (1.4). In [**7**], it was proved that they form a complete system and analytic bounds on their roots were derived.

As we will refer in section 2, the above q-trigonometric functions can be written using the Third Jackson q-Bessel funtion (or the Hahn-Exton q-Bessel function). In [**5**], analytic bounds were derived for the zeros of this function –which includes, as particular cases, the corresponding results established in [**7**]– and recently, in [**4**], it was shown that they define a complete system.

Throughout this paper we will follow the notation used in [**13**] which is now standard.

The publications [**7, 8, 9, 10, 20, 21**] are the most affiliated with this work. For other type of expansions (sampling theory) or related topics see [**1, 2, 3, 5, 6**].

2. The q-Linear Sine and Cosine. Properties.

The initial value problem

$$\frac{\delta f(x)}{\delta x} = \lambda f(x)\,, \quad f(0) = 1\,,$$

has the analytic solution [**7**]

$$\exp_q[\lambda(1-q)z] = \sum_{n=0}^{\infty}\frac{[\lambda(1-q)z]^n q^{(n^2-n)/4}}{(q;q)_n}\,, \tag{2.1}$$

which is a standard q-analog of the classical exponential function [**13, 18**]. The q-linear sine and cosine, $S_q(z)$ and $C_q(z)$, are then defined by

$$\exp_q iz := C_q(z) + iS_q(z)\,.$$

From (2.1) we get

$$C_q(z) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n[n-(1/2)]}z^{2n}}{(q;q^2;q^2)_n} \;=\; {}_1\phi_1\left(\begin{matrix}0\\ q\end{matrix}\;;\; q^2,\; q^{1/2}z^2\right)_,$$

$$S_q(z) = \frac{z}{1-q}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n[n+(1/2)]}z^{2n}}{(q^2;q^3;q^2)_n} \;=\; \frac{z}{1-q}\,{}_1\phi_1\left(\begin{matrix}0\\ q^3\end{matrix}\;;\; q^2,\; q^{3/2}z^2\right),$$

which can be written in terms of the third Jackson q-Bessel function (or, Hahn-Exton q-Bessel function) [**15, 17, 22**]

$$J_\nu^{(3)}(z;q) := z^\nu \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty} \, {}_1\phi_1\left(\begin{array}{c} 0 \\ q^{\nu+1} \end{array} ; \; q, \; qz^2\right)$$

as

$$C_q(z) = q^{-3/8} \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J^{(3)}_{-1/2}\left(q^{-3/4}z;q^2\right) ,$$

$$S_q(z) = q^{1/8} \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} z^{1/2} J^{(3)}_{1/2}\left(q^{-1/4}z;q^2\right)$$

They satisfy [**7**]

$$\frac{\delta C_q(\omega z)}{\delta z} = -\frac{\omega}{1-q} S_q(\omega z) , \tag{2.2}$$

$$\frac{\delta S_q(\omega z)}{\delta z} = \frac{\omega}{1-q} C_q(\omega z) , \tag{2.3}$$

and, when ω is such that $S_q(\omega) = 0$,

$$\left[C_q(\omega)\right]^{-1} = C_q(q^{-1/2}\omega) = C_q(q^{1/2}\omega) . \tag{2.4}$$

It is known [**7**] that the roots of $C_q(z)$ and $S_q(z)$ are *real*, *simple* and *countable*. Further, because $C_q(z)$ and $S_q(z)$ are respectively even and odd functions, the roots of $C_q(z)$ and $S_q(z)$ are symmetric and we will denote the positive zeros of the function $S_q(z)$ by ω_k, $k = 1, 2, \ldots$, with $\omega_1 < \omega_2 < \omega_3 < \ldots$.

As we mentioned before, the zeros of the function $S_q(z)$ form a discrete set of symmetric points in the real line. In [**7**, page 145], it was shown that the set of positive zeros ω_k, $k = 1, 2, \ldots$ of the function $S_q(z)$, verify the following *analytic bounds*:

If $0 < q < \beta_0$, *where* β_0 *is the root of* $(1-q^2)^2 - q^3$, $0 < q < 1$, *then*

$$q^{-k+\alpha_k+1/4} < \omega_k < q^{-k+1/4} , \quad k = 1, 2, \ldots ,$$

where

$$\alpha_k \equiv \alpha_k(q) = \frac{\log\left[1 - \frac{q^{2k+1}}{1-q^{2k}}\right]}{2\log q} , \quad k = 1, 2, \ldots .$$

According to *Remark 1* in [**7**, page 145], the previous result can be restated in the following form:

Theorem A *For every* q, $0 < q < 1$, K *exists such that if* $k \geq K$ *then*

$$\omega_k = q^{-k+\epsilon_k+1/4} , \quad 0 < \epsilon_k < \alpha_k(q) .$$

By using Taylor expansion one finds out that

$$\alpha_k(q) = \mathcal{O}(q^{2k}) \quad \text{as} \quad k \to \infty . \tag{2.5}$$

Theorem 4.1 of [**7**, page 139] settle the *orthogonality relations*:

Theorem B *Considering* $\mu_k = (1-q)C_q(q^{1/2}\omega_k)S_q'(\omega_k)$ *we have*

$$\int_{-1}^{1} C_q(q^{1/2}\omega_k x)C_q(q^{1/2}\omega_m x)d_q x = \begin{cases} 0 & \text{if} \quad k \neq m \\ 2 & \text{if} \quad k = 0 = m \\ \mu_k & \text{if} \quad k = m \neq 0 \end{cases}$$

$$\int_{-1}^{1} S_q(q\omega_k x)S_q(q\omega_m x)d_q x = \begin{cases} 0 & if \quad k \neq m \vee k = 0 = m \\ \\ q^{-1/2}\mu_k & if \quad k = m \neq 0 \end{cases} .$$

The *Completeness Theorem* [**7**, page 153], where a misprint is corrected, states the following:

Theorem C *Let* $f(\omega_k z) = C_q\left(q^{\frac{1}{2}}\omega_k z\right) + iS_q(q\omega_k z)$ *where the* ω_k , $\omega_0 = 0 < \omega_1 < \omega_2 < \ldots$ *are the non-negative roots of* $S_q(z)$. *Suppose that*

$$\int_{-1}^{1} g(z)f(\omega_k z)d_q z = 0 \quad , \qquad k = 0, 1, 2, \ldots$$

where $g(z)$ *is bounded on* $z = \pm q^j$, $j = 0, 1, 2, \ldots$. *Then,* $g(z) \equiv 0$, *i.e.,* $g\left(\pm q^j\right) = 0$ *for all* $j = 0, 1, 2, \ldots$.

To end this section we write down the Theorem 6.2 of [**7**, page 150]:

Theorem D *If* $S_q(\omega_k) = 0$ *then, for* $n = 0, 1, 2, \ldots$,

$$S_q(q^{1+n}\omega_k) = S_q(q\omega_k)\sum_{j=0}^{n}(-1)^j q^{j(j+\frac{1}{2})}\frac{\left(q^{1+n-j};q\right)_{2j+1}}{(q;q)_{2j+1}}\left(\omega_k^2\right)^j ,$$

$$C_q(q^{\frac{1}{2}+n}\omega_k) = C_q(q^{\frac{1}{2}}\omega_k)\sum_{j=0}^{n}(-1)^j q^{j(j-\frac{1}{2})}\frac{\left(q^{1+n-j};q\right)_{2j}}{(q;q)_{2j}}\left(\omega_k^2\right)^j .$$

3. The Fourier Coefficients

As a consequence of the orthogonality relations of Theorem B, we may consider formal Fourier expansions of the form

$$f(x) \sim S_q[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k C_q\left(q^{\frac{1}{2}}\omega_k x\right) + b_k S_q\left(q\omega_k x\right)\right] , \tag{3.1}$$

with $a_0 = \int_{-1}^{1} f(t)d_q t$ and, for $k = 1, 2, 3, \ldots$,

$$a_k = \frac{1}{\mu_k}\int_{-1}^{1} f(t)C_q\left(q^{\frac{1}{2}}\omega_k t\right)d_q t \tag{3.2}$$

$$b_k = \frac{q^{\frac{1}{2}}}{\mu_k}\int_{-1}^{1} f(t)S_q\left(q\omega_k t\right)d_q t , \tag{3.3}$$

where

$$\mu_k = (1-q)C_q(q^{1/2}\omega_k)S_q'(\omega_k) . \tag{3.4}$$

In order to study the convergence of the series (3.1)-(3.4), it becomes clear that we need to know the behavior of the factor μ_k of the denominator as $k \to \infty$, which is equivalent to control the behavior of $S_q'(\omega_k)$ and $C_q(q^{1/2}\omega_k)$ as $k \to \infty$.

Theorem 3.2 from [**10**] asserts that

Theorem E *At least for* $0 < q \le (1/51)^{1/50}$,

$$S_q'(\omega_k) = \frac{2}{1-q} q^{-(k-\frac{1}{2}-\epsilon_k)^2} S_k \,,$$

where S_k *satisfies* $\liminf_{k\to\infty} |S_k| > 0$.

With respect to S_k from the previous theorem we have the following lemma:

Lemma 3.1. *There exists a constant* B, *independent of* k, *such that*

$$|S_k| \le B \,, \quad k = 1, 2, 3, \ldots .$$

Proof. The expression of S_k is given [**7**, page 147] by

$$S_k = \sum_{n=0}^{\infty} \frac{(-1)^n n q^{(n-k+1/2+\varepsilon_k)^2}}{(q^2, q^3; q^2)_n} = (-1)^k \sum_{m=-k}^{\infty} \frac{(-1)^m m q^{(m+1/2+\varepsilon_k)^2}}{(q^2, q^3; q^2)_{m+k}} \,.$$

For k large enough, by Theorem A and (2.5), $1/2 + \varepsilon_k > 0$ hence

$$|S_k| \le \sum_{m=-k}^{\infty} \frac{|m| q^{(m+1/2+\varepsilon_k)^2}}{(q^2, q^3; q^2)_{m+k}} \le \frac{2}{(q^2; q)_\infty} \sum_{m=1}^{\infty} m q^{(m-1)^2} = B$$

which completes the proof since the infinite series on the right member is convergent. □

We observe that the constant B, as well as S_k, depend on the parameter q.

The behavior of $C_q(q^{1/2}\omega_k)$ as $k \to \infty$ will be known by the corresponding behavior of $C_q(\omega_k)$ and by (2.4). Theorem 3.3 of [**10**] establishes

Theorem F *At least for* $0 < q \le (1/50)^{1/49}$,

$$C_q(\omega_k) = q^{-(k-\epsilon_k)^2} R_k \,,$$

$$\textit{where} \quad |R_k| < \frac{2}{(1-q)(q;q)_\infty} \quad \textit{and} \quad \liminf_{k\to\infty} |R_k| > 0 \,.$$

To end this section, we collect the Theorems 4.1, 4.2 and 4.3 of [**10**]:

Theorem G *If* $c \in \mathbb{R}$ *exists such that, as* $k \to \infty$,

$$\int_{-1}^{1} f(t) C_q\left(q^{\frac{1}{2}} \omega_k t\right) d_q t = \mathcal{O}\left(q^{ck}\right) \quad \text{and} \quad \int_{-1}^{1} f(t) S_q\left(q \omega_k t\right) d_q t = \mathcal{O}\left(q^{ck}\right)$$

then, at least for $0 < q \le (1/51)^{1/50}$, *the* q*-Fourier series (3.1) is pointwise convergent at each fixed point* $x \in V_q = \left\{ \pm q^{n-1} : n \in \mathbb{N} \right\}$.

Theorem H *If* $c > 1$ *exists such that, as* $k \to \infty$,

$$\int_{-1}^{1} f(t) C_q\left(q^{\frac{1}{2}} \omega_k t\right) d_q t = \mathcal{O}\left(q^{ck}\right) \quad \text{and} \quad \int_{-1}^{1} f(t) S_q\left(q \omega_k t\right) d_q t = \mathcal{O}\left(q^{ck}\right)$$

then, the q-Fourier series (3.1), at least for $0<q\leq(1/51)^{1/50}$ *, converges uniformly on* $V_q=\left\{\pm q^{n-1}:\, n\in\mathbb{N}\right\}$.

Theorem I *If* f *is a bounded function on the set* $V_q=\left\{\pm q^{n-1}:\, n\in\mathbb{N}\right\}$ *, and the q-Fourier series* $S_q[f](x)$ *converges uniformly on* V_q *then its sum is* $f(x)$ *whenever* $x\in V_q$.

4. Convergence condition on the function

Denoting the q-Fourier coefficients of a function f by $a_k\bigl(f(x)\bigr)$ and $b_k\bigl(f(x)\bigr)$, $k=1,2,3,\dots$, using (3.2)-(3.4) and (2.2)-(2.3) one have, by (1.5),

$$a_k\bigl(f(x)\bigr)-\frac{1-q}{q^{1/2}\omega_k\mu_k}\int_{-1}^{1}S_q\left(q\omega_k t\right)\frac{\delta f\bigl(q^{\frac12}t\bigr)}{\delta t}d_qt-\frac{1-q}{q\omega_k}\,b_k\left(\frac{\delta f(q^{\frac12}x)}{\delta x}\right) \tag{4.1}$$

and

(4.2)

$$\begin{aligned} b_k(f(x))=\frac{q-1}{q^{\frac12}\omega_k\mu_k}\,&\Big\{q^{\frac12}\Bigl[f\bigl(q^{-1}\bigr)-f\bigl(-q^{-1}\bigr)\Bigr]C_q\left(q^{\frac12}\omega_k\right)-\\ &q^{\frac12}\Bigl[f\bigl(0^{+}\bigr)-f\bigl(0^{-}\bigr)\Bigr]-\int_{-1}^{1}C_q\left(q^{\frac12}\omega_k t\right)\frac{\delta f\bigl(q^{-\frac12}t\bigr)}{\delta t}d_qt\Big\}\\ =\frac{1-q}{q^{\frac12}\omega_k}&\left\{a_k\left(\frac{\delta f(q^{-\frac12}x)}{\delta x}\right)+q^{\frac12}\left[\frac{f\bigl(0^{+}\bigr)-f\bigl(0^{-}\bigr)}{\mu_k}-\frac{f\bigl(q^{-1}\bigr)-f\bigl(-q^{-1}\bigr)}{(1-q)S_q'(\omega_k)}\right]\right\}. \end{aligned}$$

The conjugation of this last two identities with Theorem H enables us to deduce conditions on the function f in order to guarantee uniform convergence of the corresponding Fourier series $S_q[f]$. In its statement, we will consider the notation

$$L_q^{\infty}[-1,1]=\Big\{f:\sup\left\{\,\bigl|f\bigl(\pm q^{n-1}\bigr)\bigr|:n\in\mathbb{N}\right\}<\infty\Big\}$$

and the following definition:

Definition 4.1 *If two constants* M *and* λ *exist such that*

$$\Bigl|f\bigl(\pm q^{n-1}\bigr)-f\bigl(\pm q^{n}\bigr)\Bigr|\leq Mq^{\lambda n}\,,\quad n=0,1,2,\dots, \tag{4.3}$$

then the function f *is said to be* q*-linear Hölder of order* λ .

Theorem 4.1. *If* $f\in L_q^{\infty}[-1,1]$ *is a* q*-linear Hölder function of order* $\lambda>\frac12$ *and satisfies* $f(0^+)=f(0^-)$ *then, at least for* $0<q\leq(1/50)^{1/49}$ *, the corresponding* q*-Fourier series* $S_q[f]$ *converges uniformly to* f *on the set of points* $V_q=\left\{\pm q^{n-1}:\, n\in\mathbb{N}\right\}$.

Proof. From (3.2) and (4.1) one have

$$\int_{-1}^{1}f(t)C_q\left(q^{\frac12}\omega_k t\right)d_qt=\mu_k\,a_k\bigl(f\bigr)=-\frac{1-q}{q^{1/2}\omega_k}\int_{-1}^{1}S_q\left(q\omega_k t\right)\frac{\delta f\bigl(q^{\frac12}t\bigr)}{\delta t}d_qt\,. \tag{4.4}$$

Similarly, from (3.3) and (4.2),

(4.5)
$$\int_{-1}^{1} f(t) S_q\left(q\omega_k t\right) d_q t = q^{-1/2}\mu_k\, b_k\big(f\big) =$$
$$\frac{q-1}{q\omega_k}\left\{q^{\frac{1}{2}}\left[f\big(q^{-1}\big)-f\big(-q^{-1}\big)\right]C_q\left(q^{\frac{1}{2}}\omega_k\right)-\int_{-1}^{1} C_q\left(q^{\frac{1}{2}}\omega_k t\right)\frac{\delta f\big(q^{-\frac{1}{2}}t\big)}{\delta t} d_q t\right\}.$$

By Cauchy-Schwarz inequality we have

(4.6)
$$\left|\int_{-1}^{1} S_q\left(q\omega_k t\right)\frac{\delta f\big(q^{\frac{1}{2}}t\big)}{\delta t} d_q t\right| \le \left(\int_{-1}^{1} S_q^2\left(q\omega_k t\right) d_q t\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left(\frac{\delta f\big(q^{\frac{1}{2}}t\big)}{\delta t}\right)^2 d_q t\right)^{\frac{1}{2}}$$

and

(4.7)
$$\left|\int_{-1}^{1} C_q\left(q^{\frac{1}{2}}\omega_k t\right)\frac{\delta f\big(q^{-\frac{1}{2}}t\big)}{\delta t} d_q t\right| \le \left(\int_{-1}^{1} C_q^2\left(q^{\frac{1}{2}}\omega_k t\right) d_q t\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left(\frac{\delta f\big(q^{-\frac{1}{2}}t\big)}{\delta t}\right)^2 d_q t\right)^{\frac{1}{2}}$$

Using the orthogonality relations of Theorem B we may write

$$q^{\frac{1}{2}}\int_{-1}^{1} S_q^2\left(q\omega_k t\right) d_q t = \int_{-1}^{1} C_q^2\left(q^{\frac{1}{2}}\omega_k t\right) d_q t = \mu_k = (1-q) C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)\,,$$

thus (4.6) and (4.7) become, respectively,

$$\left|\int_{-1}^{1} S_q\left(q\omega_k t\right)\frac{\delta f\big(q^{\frac{1}{2}}t\big)}{\delta t} d_q t\right| \le \tag{4.8}$$
$$q^{-\frac{1}{4}}(1-q)^{\frac{1}{2}}\left(C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left(\frac{\delta f\big(q^{\frac{1}{2}}t\big)}{\delta t}\right)^2 d_q t\right)^{1/2}$$

and

$$\left|\int_{-1}^{1} C_q\left(q^{\frac{1}{2}}\omega_k t\right)\frac{\delta f\big(q^{-\frac{1}{2}}t\big)}{\delta t} d_q t\right| \le \tag{4.9}$$
$$(1-q)^{\frac{1}{2}}\left(C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left(\frac{\delta f\big(q^{-\frac{1}{2}}t\big)}{\delta t}\right)^2 d_q t\right)^{\frac{1}{2}}.$$

Now, using the corresponding definitions of the q-integral and of the operator δ one finds that

$$\int_{-1}^{1}\left(\frac{\delta f\big(q^{\frac{1}{2}}t\big)}{\delta t}\right)^2 d_q t =$$
$$(1-q)\sum_{n=0}^{\infty}\left\{\left[f\big(q^n\big)-f\big(q^{n+1}\big)\right]^2+\left[f\big(-q^n\big)-f\big(-q^{n+1}\big)\right]^2\right\} q^{-n}$$

hence, since f is q-linear Hölder of order $\lambda > \frac{1}{2}$, by (4.3),

$$\int_{-1}^{1}\left(\frac{\delta f\left(q^{\frac{1}{2}}t\right)}{\delta t}\right)^{2} d_q t \leq 2M^2(1-q)\sum_{n=0}^{\infty} q^{(2\lambda-1)n} = \frac{2(1-q)M^2}{1-q^{2\lambda-1}}\,. \tag{4.10}$$

In a similar way we obtain

$$\int_{-1}^{1}\left(\frac{\delta f\left(q^{-\frac{1}{2}}t\right)}{\delta t}\right)^{2} d_q t \leq \frac{2(1-q)M^2}{1-q^{2\lambda-1}}\,. \tag{4.11}$$

Thus, (4.8) and (4.9) become, respectively,

$$\left|\int_{-1}^{1} S_q\left(q\omega_k t\right)\frac{\delta f\left(q^{\frac{1}{2}}t\right)}{\delta t} d_q t\right| \leq \frac{\sqrt{2}q^{-\frac{1}{4}}(1-q)M}{\sqrt{1-q^{2\lambda-1}}}\left(C_q\left(q^{\frac{1}{2}}\omega_k\right)S_q'(\omega_k)\right)^{\frac{1}{2}} \tag{4.12}$$

and

$$\left|\int_{-1}^{1} C_q\left(q^{\frac{1}{2}}\omega_k t\right)\frac{\delta f\left(q^{-\frac{1}{2}}t\right)}{\delta t} d_q t\right| \leq \frac{\sqrt{2}(1-q)M}{\sqrt{1-q^{2\lambda-1}}}\left(C_q\left(q^{\frac{1}{2}}\omega_k\right)S_q'(\omega_k)\right)^{\frac{1}{2}}\,. \tag{4.13}$$

Finally, using (4.12) and (4.13) in (4.4) and (4.5), respectively, by Theorems A, E, F and identity (2.4), as well as Lemma 3.1, one concludes that the conditions of Theorem H are fulfilled with, for instance, $c = 3/2$, thus the q-Fourier series (3.1), at least for $0 < q \leq (1/50)^{1/49}$, converges uniformly on the set $V_q = \left\{\pm q^{n-1} : n \in \mathbb{N}\right\}$, hence, by Theorem I, under the same restriction on q,

$$S_q[f](x) = f(x)\,, \quad \forall\, x \in V_q = \left\{\pm q^{n-1} : n \in \mathbb{N}\right\}\,.$$

□

A simple analysis of the previous theorem shows immediately that the behavior of the function f at the origin is crucial to study the convergence of the q-Fourier series $S_q[f]$. Consider, then, the following concept:

Definition 4.2 *A function f is said to be almost q-linear Hölder of order λ if two constants M, λ and a positive integer n_0 exist such that*

$$\left|f\left(\pm q^{n-1}\right) - f\left(\pm q^{n}\right)\right| \leq Mq^{\lambda n} \tag{4.14}$$

holds for every $n \geq n_0$.

Obviously that every *q-linear Hölder function of order λ* is *almost q-linear Hölder function of order λ*.

Corollary 4.2. *If a function $f \in L_q^{\infty}[-1,1]$ is almost q-linear Hölder of order $\lambda > \frac{1}{2}$ and satisfies $f(0^+) = f(0^-)$ then, at least for $0 < q \leq (1/50)^{1/49}$, the corresponding q-Fourier series $S_q[f]$ converges uniformly to f on the set of points $V_q = \left\{\pm q^{n-1} : n \in \mathbb{N}\right\}$.*

Proof. By hypothesis, f is almost q-linear Hölder of order $\lambda > 1/2$, i.e., it satisfies (4.14). Then the relations (4.10) and (4.11)) now become

$$\int_{-1}^{1}\left(\frac{\delta f\left(q^{\frac{1}{2}}t\right)}{\delta t}\right)^{2} d_q t \leq \frac{2(1-q)M_1^2\,q^{n_0}}{1-q^{2\lambda-1}}$$

and

$$\int_{-1}^{1}\left(\frac{\delta f\left(q^{-\frac{1}{2}}t\right)}{\delta t}\right)^2 d_qt \leq \frac{2(1-q)M_2^2\,q^{n_0}}{1-q^{2\lambda-1}}\,,$$

respectively, where M_1 and M_2 are constants. Therefore, using the above inequalities in formulas (4.8) and (4.9) we get two new inequalities that differ from (4.12) and (4.13) only by a constant in the corresponding right hand side. Hence, the conclusion on the uniform convergence follows. □

COROLLARY 4.3. *If $f \in L_q^{\infty}[-1,1]$ satisfies $f(0^+) = f(0^-)$ and there exists a neighborhood of the origin where the function f is continuous and piecewise smooth then, at least for $0 < q \leq (1/50)^{1/49}$, the corresponding q-Fourier series $S_q[f]$ converges uniformly to f on the set of points $V_q = \left\{\pm q^{n-1} : n \in \mathbb{N}\right\}$.*

PROOF. It's just a consequence of the fact that a function f that is continuous and piecewise smooth at any neighborhood of the origin satisfies a Lipschitz condition [**16**, page 204]. Thus, it satisfies a Hölder condition of order 1 on that neighborhood and so, by Corollary 4.2, the uniform convergence follows. □

5. Convergence on and outside the q-linear grid

The convergence of the basic Fourier series (3.1)-(3.4) always refer to the discrete set of the points of the q-linear grid $V_q = \left\{\pm q^{n-1} : n \in \mathbb{N}\right\}$.
Two important questions arise at this moment:

- *The above mentioned q-Fourier series also converges outside the points of the q-linear grid?*
- *In that case, to what function it converges?*

Next theorem will give a positive answer to both questions.

THEOREM 5.1. *Let $f \in L_q^{\infty}[-1,1]$ and suppose that $c \in \mathbb{R}^+$ exists such that, as $k \to \infty$,*
(5.1)
$$\int_{-1}^{1} f(t)C_q\left(q^{\frac{1}{2}}\omega_k t\right)d_qt = \mathcal{O}\left(q^{(k+c)^2}\right)\,,\ \int_{-1}^{1} f(t)S_q(q\omega_k t)\,d_qt = \mathcal{O}\left(q^{(k+c-\frac{1}{2})^2}\right).$$
If f is analytic inside $C_\delta = \{z \in \mathbb{C} : |z| < \delta\}$, where δ is a positive quantity such that $0 < \delta \leq q^{-\sigma}$ with $0 < \sigma < c$, then, at least for $0 < q \leq \sqrt[50]{1/51}$,

$$f(z) = S_q[f](z) \quad in \quad C_\delta = \{\, z \in \mathbb{C} : |z| < \delta\,\}\,. \tag{5.2}$$

PROOF. We first notice that

$$C_q\left(q^{\frac{1}{2}}\omega_k z\right) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}}{\left(q^2,q;q^2\right)_n}q^{\frac{3}{2}n}\omega_k^{2n}z^{2n}$$

and

$$S_q\left(q\omega_k z\right) = \frac{q\omega_k z}{1-q}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)}}{\left(q^2,q^3;q^2\right)_n}q^{\frac{7}{2}n}\omega_k^{2n}z^{2n}$$

hence, for sufficiently large values of k, by Theorem A, whenever $|z| \leq q^{-\sigma}$,

$$\begin{aligned} \left|C_q\left(q^{\frac{1}{2}}\omega_k z\right)\right| &\leq \sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{\left(q^2,q;q^2\right)_n}q^{2n(1-k+\epsilon_k)}\left(q^{-\sigma}\right)^{2n} \\ &\leq \frac{q^{-\left(k-\frac{1}{2}+\sigma-\epsilon_k\right)^2}}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{\left(n-k+\frac{1}{2}-\sigma+\epsilon_k\right)^2} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \left|S_q\left(q\omega_k z\right)\right| &\leq \frac{q\omega_k z}{1-q}\sum_{n=0}^{\infty}\frac{q^{n(n-1)}}{\left(q^2,q^3;q^2\right)_n}q^{2n(2-k+\epsilon_k)}\left(q^{-\sigma}\right)^{2n} \\ &\leq \frac{q^{\frac{5}{4}-k+\epsilon_k-\left(k-\frac{3}{2}+\sigma-\epsilon_k\right)^2}}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{\left(n-k+\frac{3}{2}-\sigma+\epsilon_k\right)^2}. \end{aligned} \tag{5.4}$$

An easy calculation shows that

$$\begin{aligned} \sum_{n=0}^{\infty}q^{\left(n-k+\frac{1}{2}+\epsilon_k-\sigma\right)^2} &= \sum_{n=0}^{k-1}q^{\left(n-k+\frac{1}{2}-\sigma+\epsilon_k\right)^2}+\sum_{n=k}^{\infty}q^{\left(n-k+\frac{1}{2}-\sigma+\epsilon_k\right)^2} \\ &= \sum_{m=0}^{k-1}q^{\left(m+\frac{1}{2}+\sigma-\epsilon_k\right)^2}+\sum_{m=0}^{\infty}q^{\left(m+\frac{1}{2}-\sigma+\epsilon_k\right)^2}. \end{aligned}$$

thus, if

$$|\,\sigma\,| < \frac{1}{2},$$

for sufficiently large values of k,

$$\sum_{n=0}^{\infty}q^{\left(n-k+\frac{1}{2}+\epsilon_k-\sigma\right)^2} < \sum_{m=0}^{k-1}q^{m^2}+\sum_{m=0}^{\infty}q^{m^2} < 2\sum_{m=0}^{\infty}q^m = \frac{2}{1-q}.$$

In a similar way, for a given $p \in \mathbb{N}_0$, if

$$|\,\sigma\,| < \frac{1}{2}+p \tag{5.5}$$

then, for sufficiently large values of k,

$$\sum_{n=0}^{\infty}q^{\left(n-k+\frac{1}{2}+\epsilon_k-\sigma\right)^2} < 2p+\frac{2}{1-q}. \tag{5.6}$$

With the same reasoning we get, again for sufficiently large values of k,

$$\sum_{n=0}^{\infty}q^{\left(n-k+\frac{3}{2}+\epsilon_k-\sigma\right)^2} < 2p+\frac{2}{1-q}. \tag{5.7}$$

Hence, by (5.3), (5.6) and (5.4), (5.7), we may write, respectively, for k large enough,

$$\left|C_q\left(q^{\frac{1}{2}}\omega_k z\right)\right| \leq \frac{2p(1-q)+2}{(q;q)_\infty}\,q^{-\left(k-\frac{1}{2}+\sigma-\epsilon_k\right)^2} \tag{5.8}$$

and

$$\left|S_q\left(q\omega_k z\right)\right| \leq \frac{2p(1-q)+2}{(q;q)_\infty}\,q^{\frac{5}{4}-k+\epsilon_k-\left(k-\frac{3}{2}+\sigma-\epsilon_k\right)^2}. \tag{5.9}$$

This way, for k large enough, using (3.2) and (3.4), Theorems E and F, relation (2.4) and inequality (5.8), at least for $0 < q \leq \sqrt[50]{1/51}$,

$$\left|a_k C_q\left(q^{\frac{1}{2}}\omega_k z\right)\right| \leq \frac{2p(1-q)+2}{(1-q)^2(q;q)_\infty^2}\left|\int_{-1}^{1} f(t)C_q\left(q^{\frac{1}{2}}\omega_k t\right) d_q t\right| \frac{q^{-\left(k-\frac{1}{2}+\sigma-\epsilon_k\right)^2-k+\frac{1}{4}+\epsilon_k}}{|S_k|}.$$

By hypothesis (5.1), we may suppose that $c_1 \in \mathbb{R}^+$ and $M_1 > 0$ exist such that, for k large enough,

$$\left|\int_{-1}^{1} f(t)C_q\left(q^{\frac{1}{2}}\omega_k t\right) d_q t\right| \leq M_1 q^{(k+c_1)^2}\,. \tag{5.10}$$

In that case we have

$$\left|a_k C_q\left(q^{\frac{1}{2}}\omega_k z\right)\right| \leq 2M_1 \frac{p(1-q)+1}{(1-q)^2(q;q)_\infty^2} \frac{q^{(k+\frac{c_1+\sigma}{2}-\frac{1}{4}-\frac{\epsilon_k}{2})(1+2(c_1-\sigma)+2\epsilon_k)-k+\frac{1}{4}+\epsilon_k}}{|S_k|}$$

hence, if $1+2(c_1-\sigma) > 1$, i.e., if $\sigma < c_1$ then, taking into account Theorem A and (2.5), and the Theorems E and F, at least for $0 < q \leq \sqrt[50]{1/51}$,

$$\left|a_k C_q\left(q^{\frac{1}{2}}\omega_k z\right)\right| \leq A_1 q^{\theta_1 k} \quad , \tag{5.11}$$

where A_1 and θ_1 are positive constants.

Analogously, for k large enough, (3.3) and (3.4), Theorems E and F, relation (2.4) and inequality (5.9),

$$\left|b_k S_q\left(q\omega_k z\right)\right| \leq \frac{2p(1-q)+2}{(1-q)^2(q;q)_\infty^2}\left|\int_{-1}^{1} f(t)S_q\left(q\omega_k t\right) d_q t\right| \frac{q^{-\left(k-\frac{3}{2}+\sigma-\epsilon_k\right)^2-2k+2+2\epsilon_k}}{|S_k|}$$

so, again by hypothesis (5.1), if we admit that $c_2 \in \mathbb{R}^+$ and $M_2 > 0$ exist such that

$$\left|\int_{-1}^{1} f(t)S_q\left(q\omega_k t\right) d_q t\right| \leq M_2 q^{(k+c_2-\frac{1}{2})^2}\,, \tag{5.12}$$

then,

$$\left|b_k S_q\left(q\omega_k z\right)\right| \leq 2M_2 \frac{p(1-q)+1}{(1-q)^2(q;q)_\infty^2} \frac{q^{(k+\frac{c_2+\sigma}{2}-\frac{3}{4}-\frac{\epsilon_k}{2})(2+2(c_2-\sigma)+2\epsilon_k)-2k+2+2\epsilon_k}}{|S_k|}\,.$$

Similarly, if $2+2(c_2-\sigma) > 2$, i.e., if $\sigma < c_2$ then, at least for q such that $0 < q \leq \sqrt[50]{1/51}$,

$$\left|b_k s_q\left(q\omega_k z\right)\right| \leq A_2 q^{\theta_2 k}\,, \tag{5.13}$$

being A_2 and θ_2 positive constants.

We remark that in (5.5) we may choose p sufficiently large in order that one haves

$$-\frac{1}{2}-p \,<\, 0 \,<\, \sigma \,<\, \min\{c_1, c_2\} \,\leq\, \frac{1}{2}+p\,, \tag{5.14}$$

thus, replacing c_1 and c_2 from (5.10) and (5.12) by $c = \min\{c_1, c_2\}$, respectively, we conclude, through (5.11) and (5.13), that the conditions (5.1) guaranty the uniform convergence of the q-Fourier series (3.1) in $C_{q^{-\sigma}} = \{\, z \in \mathbb{C} \,:\, |z| \,<\, q^{-\sigma} \,\}$ if σ satisfies (5.14). This way, under this condition on σ, we have, by Theorem H,

$$f(x) = S_q[f](x) \quad \text{whenever} \quad x \in V_q\,,$$

since $V_q \subset C_{q^{-\sigma}}$, where $V_q = \left\{ q^{n-1} : n \in \mathbb{N} \right\}$ is the corresponding set of Theorem I and $C_{q^{-\sigma}}$ is the interior of the circle of the complex plane with center at the origin and radius $q^{-\sigma}$.

On the other side, again by the uniform convergence of the q-Fourier series $S_q[f](x)$ on $C_{q^{-\sigma}}$, since the terms of the mentioned q-Fourier series are entire functions we then have that the q-series is analytic inside $C_{q^{-\sigma}}$. From the continuity of both members of the above equality it results $f(0) = S_q[f](0)$. Thus, if f is analytic inside $C_\delta = \{ z \in \mathbb{C} : |z| < \delta \}$, where $0 < \delta \le q^{-\sigma}$, then $f(z)$ and $S_q[f](z)$ are analytic inside C_δ and coincide in a set with a limit point in the interior of such circle; by the *principle of analytic continuation* [**11**, Corollary 4.4.1], the above mentioned functions must coincide in the whole set C_δ , which proves (5.2).

□

6. Examples

In this section we will present four examples of q-Fourier series and study the corresponding questions about convergence.

Example 1: $g(x) = |x|$

The basic Fourier series of the absolute value function is given [**10**] by

$$S_q[g](x) = \frac{1}{1+q} - 2q^{-\frac{1}{2}}(1-q)\sum_{k=1}^{\infty} \frac{1 - C_q\left(q^{\frac{1}{2}}\omega_k\right)}{\omega_k^2 C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)} C_q\left(q^{\frac{1}{2}}\omega_k x\right) .$$

Conditions of Theorem H are fulfilled [**10**] with, for instance, $c = 2$. Thus, at least for $0 < q \le (1/50)^{1/49}$, the q-Fourier series of the function $f(x) = |x|$ converges uniformly on the set $V_q = \left\{ \pm q^{n-1} : n \in \mathbb{N} \right\}$ so, under the same restrictions on q , by Theorem I,

$$|x| \;=\; \frac{1}{1+q} - 2q^{-\frac{1}{2}}(1-q)\sum_{k=1}^{\infty} \frac{1 - C_q\left(q^{\frac{1}{2}}\omega_k\right)}{\omega_k^2 C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)} C_q\left(q^{\frac{1}{2}}\omega_k x\right)$$

for all $x \in V_q = \left\{ \pm q^{n-1} : n \in \mathbb{N} \right\}$.

Now, we may obtain the same conclusion in a easier way through Theorem 4.1, by simple arguing that the absolute value function

- is bounded on $V_q = \left\{ \pm q^{n-1} : n \in \mathbb{N} \right\}$,
- is continuous at the origin,
- and satisfies the q-linear Hölder condition of order 1 since

$$\Big| \left| \pm q^{n-1} \right| - \left| \pm q^{n} \right| \Big| \le (1-q)q^{n-1} .$$

Thus, by Theorem 4.1, the same conclusion over the uniform convergence follows. Notice that Corollaries 4.2 or 4.3 also apply.

Given a function f , it is important to point out that Theorem 4.1 or its Corollaries 4.2 and 4.3, enable one to decide over the uniform convergence of the q-Fourier series $S_q[f]$ without the need to compute the corresponding coefficients: only requires a short study of the function itself.

Example 2: $$h(x) = \begin{cases} -1 & \text{if} \quad x \leq 0 \\ \\ 1 & \text{if} \quad x > 0 \end{cases}$$

In this example, the conditions of Theorem H were not satisfied [**10**, Remark 3]. It was shown, using Theorem G, that the q-Fourier series

$$S_q[h](x) = 2\sum_{k=1}^{\infty} \frac{1 - C_q\left(q^{\frac{1}{2}}\omega_k\right)}{\omega_k C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)} S_q(q\omega_k x)$$

is (pointwise) convergent at each (fixed) point $x \in V_q$. Theorem 4.1 doesn't apply too (neither its corollaries) since $h(0^+) \neq h(0^-)$.

Example 3: $$H^{(a)}(x) = \begin{cases} -1 & \text{se} \quad x \leq a \\ \\ 1 & \text{se} \quad x > a \end{cases} \quad ; \qquad (a > 0)$$

Once $0 < q < 1$ is fixed, denote by n_a the least positive integer j such that $q^j < a$, i.e., $n_a = \left[\log_q a\right] + 1$. Then

$$a_0 = -2q^{n_a} \tag{6.1}$$

and, for $k = 1, 2, 3, \ldots$,

$$a_k = \frac{2(1-q)}{q^{-\frac{1}{2}+n_a}\omega_k^2\mu_k}\left[C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right) - C_q\left(q^{-\frac{1}{2}+n_a}\omega_k\right)\right].$$

By Theorem D,

$$C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right) - C_q\left(q^{-\frac{1}{2}+n_a}\omega_k\right) = q^{-\frac{1}{2}+n_a}\omega_k S_q\left(q^{n_a}\omega_k\right),$$

thus

$$a_k = -\frac{2(1-q)S_q\left(q^{n_a}\omega_k\right)}{\omega_k\mu_k} = -\frac{2}{\omega_k}\frac{S_q\left(q^{n_a}\omega_k\right)}{C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)}. \tag{6.2}$$

For $k = 1, 2, 3, \ldots$ we have

$$b_k = -\frac{2(1-q)}{\omega_k^2\mu_k}\left[\frac{S_q\left(q^{1+n_a}\omega_k\right) - S_q\left(q^{n_a}\omega_k\right)}{q^{n_a}} - S_q(q\omega_k)\right].$$

By Theorem D,

$$S_q\left(q^{1+n_a}\omega_k\right) - S_q\left(q^{n_a}\omega_k\right) = -q^{n_a}\,\omega_k\,C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right),$$

so, by (2.4),

$$b_k = \frac{2(1-q)}{\omega_k\mu_k}\left[C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right) - C_q\left(q^{\frac{1}{2}}\omega_k\right)\right] = \frac{2}{\omega_k}\frac{C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right) - C_q\left(q^{\frac{1}{2}}\omega_k\right)}{C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)}. \tag{6.3}$$

hence, substituting (6.1), (6.2) and (6.3) into (3.1) it becomes

(6.4)

$$S_q[H^{(a)}](x) = -q^{n_a} - 2\sum_{k=1}^{\infty} \frac{S_q\left(q^{n_a}\omega_k\right) C_q\left(q^{\frac{1}{2}}\omega_k x\right) + \left[C_q\left(q^{\frac{1}{2}}\omega_k\right) - C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right)\right] S_q(q\omega_k x)}{\omega_k\, C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)}\,.$$

We notice that *Example 2* follows from *Example 4* by computing the limit $n_a \to \infty$, i.e., when $a \to 0$. Again by Theorem D,

$$S_q(q^{n_a}\omega_k) = S_q(q\omega_k)\sum_{j=0}^{n_a-1}(-1)^j q^{j(j+\frac{1}{2})}\frac{\left(q^{n_a-j};q\right)_{2j+1}}{(q;q)_{2j+1}}\omega_k^{2j}$$

and

$$C_q(q^{\frac{1}{2}+n_a}\omega_k) = C_q(q^{\frac{1}{2}}\omega_k)\sum_{j=0}^{n_a}(-1)^j q^{j(j-\frac{1}{2})}\frac{\left(q^{1+n_a-j};q\right)_{2j}}{(q;q)_{2j}}\omega_k^{2j}\,,$$

thus, since $S_q(q\omega_k) = -\omega_k C_q\left(q^{1/2}\omega_k\right)$, for $k = 1, 2, 3, \dots$,

$$\int_{-1}^{1} H^{(a)}(x) C_q(q^{\frac{1}{2}}\omega_k x) d_q t = 2(1-q) C_q\left(q^{\frac{1}{2}}\omega_k\right)\sum_{j=0}^{n_a-1}(-1)^j q^{j(j+\frac{1}{2})}\frac{\left(q^{n_a-j};q\right)_{2j+1}}{(q;q)_{2j+1}}\omega_k^{2j}$$

and

$$\int_{-1}^{1} H^{(a)}(x) S_q(q\omega_k x) d_q t = 2q^{-\frac{1}{2}}(1-q)\frac{c_q\left(q^{\frac{1}{2}}\omega_k\right)}{\omega_k}\times \left[\sum_{j=0}^{n_a}(-1)^j q^{j(j-\frac{1}{2})}\frac{\left(q^{1+n_a-j};q\right)_{2j}}{(q;q)_{2j}}\omega_k^{2j} - 1\right]\,.$$

For each fixed $a > 0$, at least for $0 < q \leq (1/50)^{1/49}$, the q-Fourier series (6.4) converges uniformly on the set $V_q = \left\{\pm q^{n-1} : n \in \mathbb{N}\right\}$: in fact, after some computations, one verifies that the conditions of Theorem H are satisfied with, for instance, $c = 2$, hence, whenever $x \in V_q$ and under the above restriction on q, we may write by Theorem I,

(6.5)

$$H^{(a)}(x) \equiv -q^{n_a} - 2\sum_{k=1}^{\infty} \frac{S_q\left(q^{n_a}\omega_k\right) C_q\left(q^{\frac{1}{2}}\omega_k x\right) + \left[C_q\left(q^{\frac{1}{2}}\omega_k\right) - C_q\left(q^{\frac{1}{2}+n_a}\omega_k\right)\right] S_q(q\omega_k x)}{\omega_k C_q\left(q^{\frac{1}{2}}\omega_k\right) S_q'(\omega_k)}\,.$$

Another approach is the following: one easily check that $H^{(a)} \in L_q^{\infty}[-1,1]$, $H^{(a)}\left(0^+\right) = 0 = H^{(a)}\left(0^-\right)$ and $H^{(a)}$ is almost q-linear Hölder of order bigger then $\frac{1}{2}$ since

$$\left|H^{(a)}\left(\pm q^{n-1}\right) - H^{(a)}\left(\pm q^{n}\right)\right| = 0\,, \quad n \geq n_a + 1 = \left[\log_q a\right] + 2\,.$$

By Corollary 4.2, the q-Fourier series $S_q\left[H^{(a)}\right]$ converges uniformly on the set V_q, thus (6.5) follows.

Example 4: $f(x) = x^m$

In [**10**, Proposition 6.1] it was presented the Fourier expansion of the function $f(x) = x^m$, $m = 0, 1, 2, \dots$, in terms of the functions C_q and S_q :

$$\begin{aligned} S_q[x^m](x) \;=\; & \frac{1+(-1)^m}{2}\frac{1-q}{1-q^{m+1}} + \\ & (q;q)_m \sum_{k=1}^{\infty} \left\{ \frac{1+(-1)^m}{S_q'(\omega_k)} \sum_{i=0}^{\left[\frac{m-2}{2}\right]} \frac{(-1)^i q^{(i+1)(i-m+\frac{1}{2})}}{\omega_k^{2i+2}(q;q)_{m-1-2i}} C_q(q^{\frac{1}{2}}\omega_k x) + \right. \\ & \left. q^{\frac{1}{2}} \frac{(-1)+(-1)^m}{S_q'(\omega_k)} \sum_{i=0}^{\left[\frac{m-1}{2}\right]} \frac{(-1)^i q^{(i+1)(i-m-\frac{1}{2})}}{\omega_k^{2i+1}(q;q)_{m-2i}} S_q(q\omega_k x) \right\} , \end{aligned}$$

where $[x]$ denotes the greatest integer which does not exceed x and we will take as zero a sum where the superior index is less than the inferior one.
Furthermore, it was proved that the conditions of Theorem H are fulfilled with , for instance, $c = 2$. Thus, at least for $0 < q \le (1/50)^{1/49}$, the q-Fourier series of the function $f(x) = x^m$ converges uniformly on the set $V_q = \left\{ \pm q^{n-1} : n \in \mathbb{N} \right\}$, so, by Theorem I,

$$x^m \;=\; S_q[x^m](x) \quad \text{whenever} \quad x \in V_q = \left\{ \pm q^{n-1} : n \in \mathbb{N} \right\} .$$

We notice that the conditions of Theorem 4.1 are trivial checked when $f(x) = x^m$.

Now, since f satisfies the conditions of Theorem 5.1 with, for instance, $c = 1$ and f is an entire function then, by Theorem 5.1,

$$S_q[x^m](x) = x^m \,, \quad \forall x \in C_\delta = \{ z \in \mathbb{C} : |z| < \delta \}$$

where $0 < \delta < q^{-\sigma}$ and $0 < \sigma < 1$.

Concluding remarks. We notice that Theorem 4.1 or Corollaries 4.2 and 4.3 are q-analogs of the corresponding classical theorems on uniform convergence for trigonometric Fourier series. See, for instance, Theorem 1 of [**16**, page 204] or Theorem 55 of [**14**, page 41].

Mathematica© suggests that Theorems (4.1) and (5.1) remain valid for $0 < q < 1$. It's an open question and to prove it a different technic is required.

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